1. Let's build a $4^{\text{th}}$ order polynomial whose first derivative goes to zero at $\alpha$, $\beta$, and $\delta$. One way to do this is let $V'(x) = (x-\alpha)(x-\beta)(x-\delta)$. Assume that $\alpha < \beta < \delta$.
   (a) What is the polynomial expansion (i.e. expand the brackets) of $V'(x)$?
   (b) Integrate your result from part (a) to determine a potential energy $V(x)$.
   (c) What are the conditions on $\alpha$, $\beta$, and $\delta$ that will guarantee $V(x)$ has two distinct local minima?
   (d) For the remaining questions, let's look at the specific case of $\alpha = 2$, $\beta = 3$, and $\delta = 5$. Plot the function $V(x)$. Your x-range should cover x=1 to x=6 to see the minima clearly.
   (e) Compute the Taylor expansion of $V(x)$ in the neighbourhood of $x = 2$. Call this expansion $V_\alpha(x)$.
   (f) Compute the Taylor expansion of $V(x)$ in the neighbourhood of $x = 3$. Call this expansion $V_\beta(x)$.
   (g) Compute the Taylor expansion of $V(x)$ in the neighbourhood of $x = 5$. Call this expansion $V_\delta(x)$.
   (h) We now consider a *small oscillation* centred around $x = \alpha$. Your taylor expansion should have terms involving $(x-\alpha)^n$. A small oscillation means $x \approx \alpha$. This means we are taking small numbers and multiplying them by *themselves* $n$ times. This gives an even smaller number. It is reasonable to assume that terms involving $n > 2$, e.g. $(x-\alpha)^3$ and above can safely be ignored. Remove these terms from your expression for $V_\alpha(x)$.
   (i) Do the same for $V_\beta(x)$ and $V_\delta(x)$.
   (j) Having made this approximation, you should have three functions, each of which is a familiar parabola. For each function $V_\alpha(x)$, $V_\beta(x)$, and $V_\delta(x)$, write an expression for the force $F_\alpha(x)$, $F_\beta(x)$, and $F_\delta(x)$.
   (k) Use your expressions for the force in the neighbourhoods of $x = \alpha$, $x = \beta$, and $x = \delta$ to write down equations of motion for a particle of mass $m$. You should have three equations of motions. Each is valid only in their own neighbourhood.
   (l) What is the frequency a particle with a small oscillation around $x = \alpha$?
   (m) What is the frequency a particle with a small oscillation around $x = \beta$?
   (n) What is the frequency a particle with a small oscillation around $x = \delta$?
   (o) Justify on physical grounds your result from (m)
   (p) Students generally wonder how small of an amplitude does an oscillation need to be in order for the above approximations to be valid. Let's answer a different question: for this system, when are the oscillations large enough for the approximation to be invalid? Does the starting point matter?
   (q) If in part (h) we had been more cautious and thrown away terms involving $n > 1$, what would have happened? You need not redo all the questions. Just look at your expression for the force.